CSx25: Digital Signal Processing
NCS224: Signals and Systems

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Outline

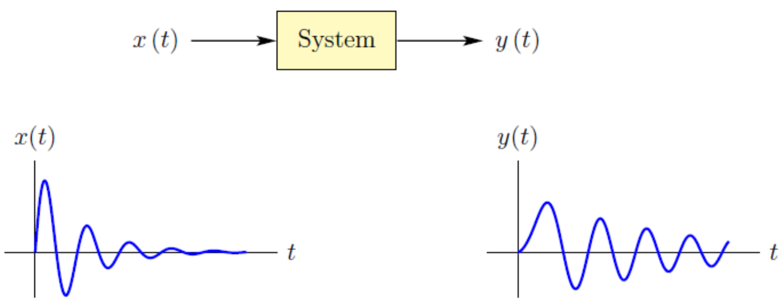
- ~~Digital Signal Processing Introduction~~
- ~~Mathematical modeling~~
- ~~Continuous Time Signals~~
- ~~Discrete Time Signals~~
- **Analyzing Continuous-Time Systems in the Time Domain**

2.1 Intro to System

Introduction

System

In general, a system is any physical entity that takes in a set of one or more physical signals and, in response, produces a new set of one or more physical signals.



A system can be viewed as any physical entity that defines the cause-effect relationships between a set of signals known as *inputs* and another set of signals known as *outputs*.

Mathematical modeling

The mathematical model of a system is a function, formula or algorithm (or a set of functions, formulas, algorithms) to approximately recreate the same cause-effect relationship between the mathematical models of the input and the output signals.

2.1 Intro to System

Consider a ***microphone*** that senses the variations in *air pressure* created by the voice of a singer and produces a small *electrical signal* in the form of a time-varying voltage. The microphone acts as a system that facilitates the conversion of an *acoustic signal* to an *electrical signal*.

Next, consider an ***amplifier*** that is connected to the output terminals of the microphone. It takes the *small-amplitude electrical signal* from the microphone and produces a *larger-scale replica* of it suitable for use with a loudspeaker.

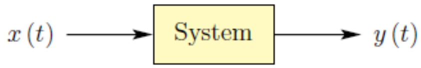
Finally, a ***loudspeaker*** that is connected to the output terminals of the amplifier converts the *electrical signal* to *sound*.

We can view each of the physical entities, namely the microphone, the amplifier and the loudspeaker, as individual systems.

Alternatively, we can look at the combination of ***all three components*** as one system that consists of three subsystems working together.

2.1 Intro to System

Introduction (continued)



$$y(t) = \text{Sys}\{x(t)\}$$

Some examples:

$$y(t) = K\,x(t)$$

$$y(t) = x(t - \tau)$$

$$y(t) = K\,[x(t)]^2$$

2.2 Linearity and Time Invariance

Linearity in continuous-time systems

Conditions for linearity

$$\text{Sys}\{x_1(t) + x_2(t)\} = \text{Sys}\{x_1(t)\} + \text{Sys}\{x_2(t)\}$$ additivity rule

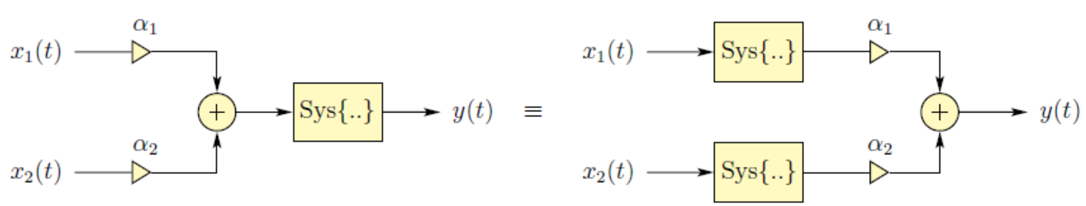
$$\text{Sys}\{\alpha_1\, x_1(t)\} = \alpha_1\ \text{Sys}\{x_1(t)\}$$ Homogeneity rule

$x_1(t)$, $x_2(t)$: Any two input signals; α_1: Arbitrary constant gain factor.

Superposition principle (combine the two conditions into one)

$$\text{Sys}\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\} = \alpha_1\ \text{Sys}\{x_1(t)\} + \alpha_2\ \text{Sys}\{x_2(t)\}$$

$x_1(t)$, $x_2(t)$: Any two input signals; α_1, α_2: Arbitrary constant gain factors.

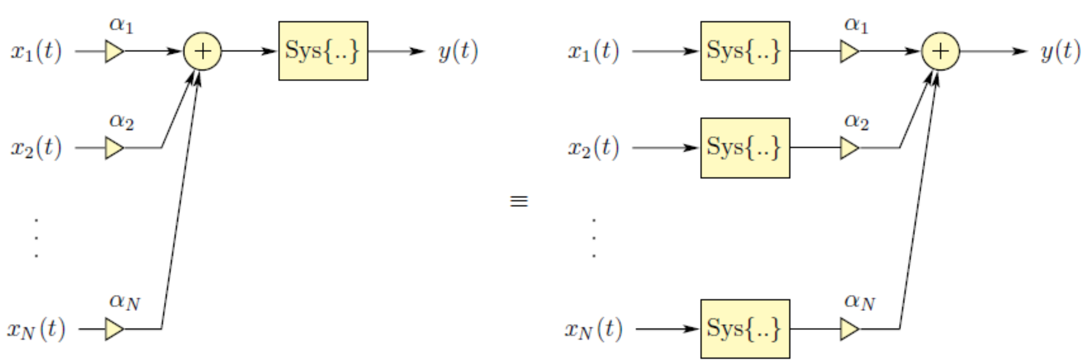


2.2 Linearity and Time Invariance

Linearity in continuous-time systems (continued)

If superposition works for the weighted sum of any two input signals, it also works for an arbitrary number of input signals.

$$\text{Sys}\left\{\sum_{i=1}^{N}\alpha_i\, x_i(t)\right\} = \sum_{i=1}^{N}\alpha_i\ \text{Sys}\{x_i(t)\} = \sum_{i=1}^{N}\alpha_i\, y_i(t)$$



A continuous-time system is *linear* if it satisfies the ***superposition principle***

2.2 Linearity and Time Invariance

Example 2.1

Testing linearity of continuous-time systems

Four different systems are described below. For each, determine if the system is linear or not:

a. $y(t) = 5\,x(t)$

b. $y(t) = 5\,x(t) + 3$

c. $y(t) = 3\,[x(t)]^2$

d. $y(t) = \cos(x(t))$

Solution:

a.

$$\begin{aligned} y(t) &= 5\,x(t) \\ &= 5\,[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)] \\ &= \alpha_1\,[5\,x_1(t)] + \alpha_2\,[5\,x_2(t)] \\ &= \alpha_1\,y_1(t) + \alpha_2\,y_2(t) \end{aligned}$$

Superposition principle holds; therefore this system is linear.

b.

$$\begin{aligned} y(t) &= 5\,x(t) + 3 \\ &= 5\,\alpha_1\,x_1(t) + 5\,\alpha_2\,x_2(t) + 3 \end{aligned}$$

Superposition principle does not hold true. The system in part (b) is not linear.

c.

$$\begin{aligned} y(t) &= 3\,[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)]^2 \\ &= 3\alpha_1^2\,[x_1(t)]^2 + 6\alpha_1\alpha_2 x_1(t)\,x_2(t) \\ &\quad + 3\alpha_2^2\,[x_2(t)]^2 \end{aligned}$$

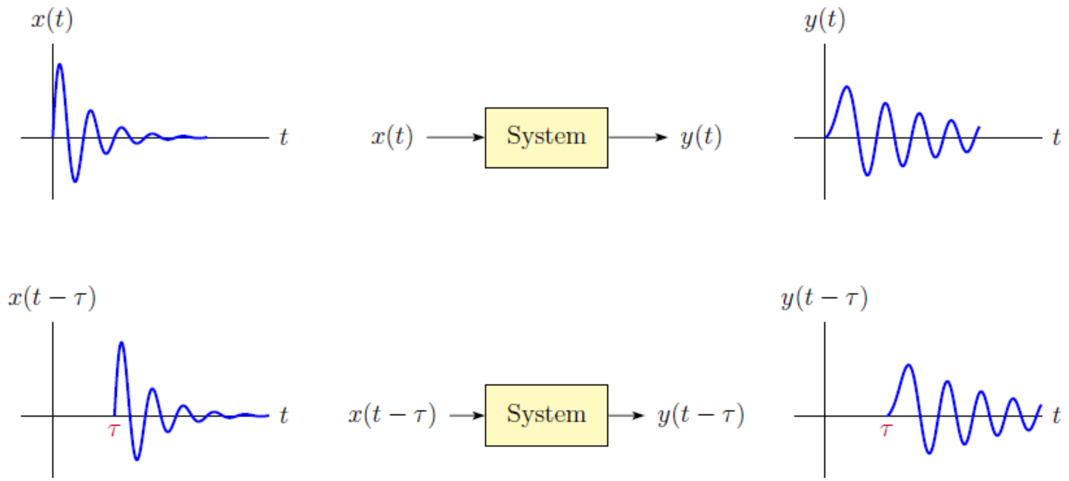
Superposition principle does not hold true. The system in part (c) is not linear.

2.2 Linearity and Time Invariance

Time-invariance in continuous-time systems

Condition for time-invariance

$$\text{Sys}\{x(t)\} = y(t) \quad \text{implies that} \quad \text{Sys}\{x(t-\tau)\} = y(t-\tau)$$

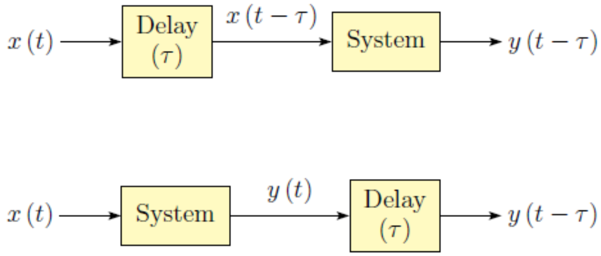


A system is said to be *time-invariant* if its behavior characteristics ***do not change in time***.

2.2 Linearity and Time Invariance

Time-invariance in continuous-time systems (continued)

Alternatively, time invariance can be explained by the equivalence of the two system configurations shown:



2.2 Linearity and Time Invariance

Example 2.2

Testing time-invariance of continuous-time systems

Three different systems are described below. For each, determine if the system is time-invariant or not:

a. $y(t) = 5\,x(t)$

b. $y(t) = 3\cos(x(t))$

c. $y(t) = 3\cos(t)\,x(t)$

<u>Solution:</u>

a. $\text{Sys}\{x(t-\tau)\} = 5\,x(t-\tau) = y(t-\tau)$ Time-invariant.

b. $\text{Sys}\{x(t-\tau)\} = 3\cos(\,x(t-\tau)\,) = y(t-\tau)$ Time-invariant.

c. $\text{Sys}\{x(t-\tau)\} = 3\cos(t)\,x(t-\tau) \neq y(t-\tau)$ Not time-invariant.

▸ MATLAB Exercise 2.2

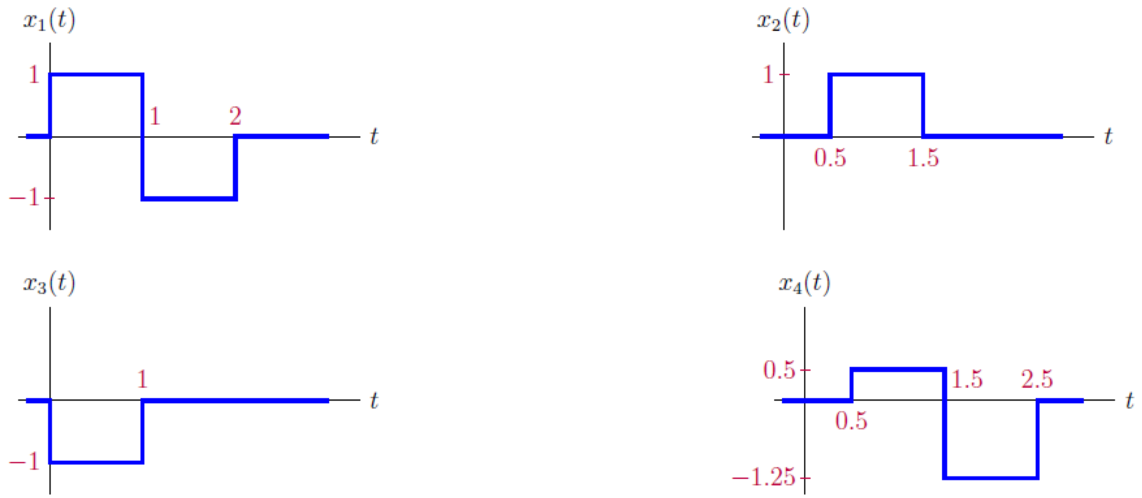
For simplicity, we will use the acronym ***CTLTI*** to refer to <u>*continuous-time linear and time-invariant*</u> systems.

2.2 Linearity and Time Invariance

Example 2.3

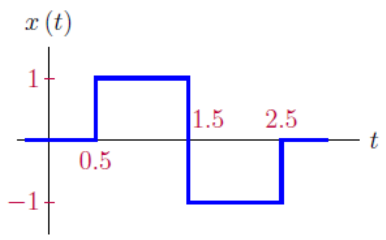
Using linearity property

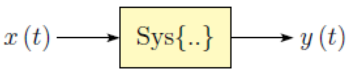
A continuous-time system is known to be linear. Whether the system is time-invariant or not is not known. Assume that the responses of the system to four input signals $x_1(t)$, $x_2(t)$ $x_3(t)$ and $x_4(t)$ shown below are known. Discuss how the information provided can be used for finding the response of this system to the signal $x(t)$ shown.



2.2 Linearity and Time Invariance

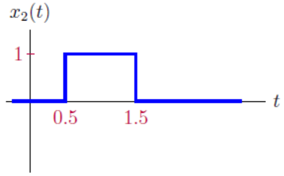
Example 2.3 (continued)

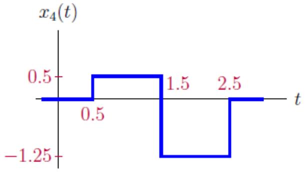




<u>Solution:</u>

$$x(t) = 0.6\,x_2(t) + 0.8\,x_4(t) \qquad \Rightarrow \qquad y(t) = 0.6\,y_2(t) + 0.8\,y_4(t)$$





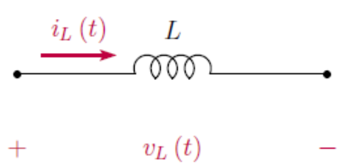
2.3 Differential Equations for Continuous-Time Systems

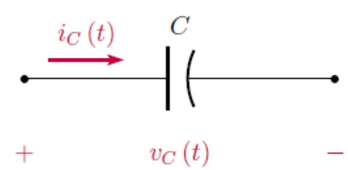
Differential equations for continuous-time systems

Example:

$$\frac{d^2y}{dt^2} + 3x(t)\,\frac{dy}{dt} + y(t) - 2x(t) = 0$$

Many physical components have mathematical models that involve integral and differential relationships between signals:





Ideal inductor:

$$v_L(t) = L\,\frac{di_L(t)}{dt}$$

Ideal capacitor:

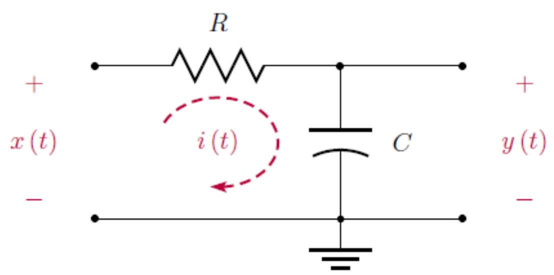
$$i_C(t) = C\,\frac{dv_C(t)}{dt}$$

2.3 Differential Equations for Continuous-Time Systems

Example 2.4

Differential equation for simple RC circuit

Find a differential equation to describe the input-output relationship for the first-order RC circuit shown.



Solution:

We know that

$$v_R(t) = R\,i(t) \quad \text{and} \quad i(t) = C\,\frac{dy(t)}{dt}$$

Use KVL to obtain

$$RC\,\frac{dy(t)}{dt} + y(t) = x(t) \qquad \Rightarrow \qquad \frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

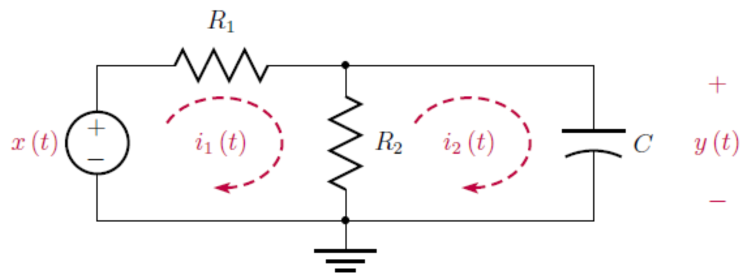
The ***order*** of a differential equation is determined by the highest-order derivative that appears in it.

2.3 Differential Equations for Continuous-Time Systems

Example 2.5

Another RC circuit

Find a differential equation to describe the input-output relationship for the first-order RC circuit shown.



Solution:

Apply KVL:

$$-x(t) + R_1\, i_1(t) + R_2\, [i_1(t) - i_2(t)] = 0$$

$$R_2\, [i_2(t) - i_1(t)] + y(t) = 0$$

$$i_2(t) = C\,\frac{dy(t)}{dt}\,, \qquad i_1(t) = C\,\frac{dy(t)}{dt} + \frac{1}{R_2}\,y(t)$$

$$-x(t) + R_1 C\,\frac{dy(t)}{dt} - \frac{R_1 + R_2}{R_2}\,y(t) = 0$$

Rearrange terms

$$\frac{dy(t)}{dt} + \frac{R_1 + R_2}{R_1 R_2 C}\,y(t) = \frac{1}{R_1 C}\,x(t)$$

2.3 Differential Equations for Continuous-Time Systems

- KVL → Circuit $x_1(t), R_1, R_2$

$$x(t) = R_1 i_1(t) + R_2 (i_1(t) - i_2(t)) \quad \text{---} \quad (1)$$

- KVL → Circuit $R_2, y(t)$

$$y(t) = R_2 (i_1(t) - i_2(t)) \quad \text{---} \quad (2)$$

- Capictor

$$i_2(t) = C \frac{dy(t)}{dt} \quad \text{---} \quad (3)$$

- Sub (3) in (2)

$$y(t) = R_2 i_1(t) - R_2 C \frac{dy(t)}{dt}$$

$$i_1(t) = \frac{1}{R_2} y(t) + C \frac{dy(t)}{dt} \quad \text{---} \quad (4)$$

- Sub (3), (4) in (1)

$$x(t) = (R_1 + R_2) i_1(t) - R_2 i_2(t)$$

$$x(t) = (R_1 + R_2)\left(\frac{1}{R_2} y(t) + C \frac{dy(t)}{dt}\right) - R_2 C \frac{dy(t)}{dt}$$

$$x(t) = \frac{(R_1 + R_2)}{R_2} y(t) + R_1 C \frac{dy(t)}{dt} + \cancel{R_2 C \frac{dy(t)}{dt}} - \cancel{R_2 C \frac{dy(t)}{dt}}$$

$$\frac{1}{R_1 C} x(t) = \frac{(R_1 + R_2)}{R_2 R_1 C} y(t) + \frac{dy(t)}{dt}$$

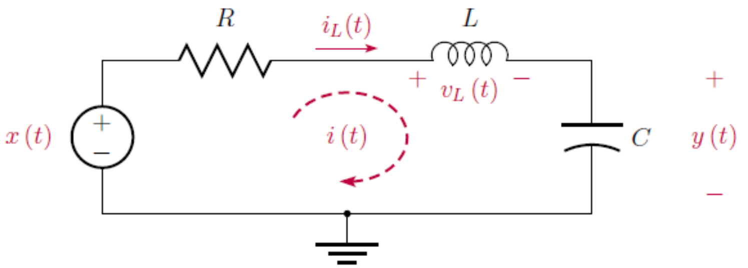
$$\boxed{\frac{dy(t)}{dt} + \frac{(R_1 + R_2)}{R_1 R_2 C} y(t) = \frac{1}{R_1 C} x(t)}$$

2.3 Differential Equations for Continuous-Time Systems

Example 2.6

Differential equation for RLC circuit

Find a differential equation to describe the input-output relationship for the RLC circuit shown.



Solution:

Apply KVL:

$$-x(t) + R\,i(t) + v_L(t) + y(t) = 0$$

$$i(t) = C\,\frac{dy(t)}{dt}\,, \qquad v_L(t) = L\,\frac{di(t)}{dt} = LC\,\frac{d^2y(t)}{dt^2}$$

$$-x(t) + RC\,\frac{dy(t)}{dt} + LC\,\frac{d^2y(t)}{dt^2} + y(t) = 0$$

Rearrange terms:

$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\,\frac{dy(t)}{dt} + \frac{1}{LC}\,y(t) = \frac{1}{LC}\,x(t)$$

2.3 Differential Equations for Continuous-Time Systems

In general, CTLTI systems can be modeled with ordinary differential equations that have constant coefficients. The differential equation that represents a CTLTI system contains the input signal *x* (*t*), the output signal *y* (*t*) as well as simple time derivatives of the two, namely

$$\frac{d^k y(t)}{dt^k}, \quad k = 0, \ldots, N$$

and

$$\frac{d^k x(t)}{dt^k}, \quad k = 0, \ldots, M$$

A general constant-coefficient differential equation representing a CTLTI system is therefore in the form

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 \, y(t) =$$
$$b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \ldots + b_1 \frac{dx(t)}{dt} + b_0 \, x(t) \qquad (2.29)$$

2.4 Constant-Coefficient Ordinary Differential Equations

Constant-coefficient ordinary differential equations

General constant-coefficient differential equation for a CTLTI system:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \ldots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

Constant-coefficient ordinary differential equation in closed summation form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Initial conditions:

$$y(t_0)\ , \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0}\ , \quad \ldots, \quad \left.\frac{d^{N-1} y(t)}{dt^{N-1}}\right|_{t=t_0}$$

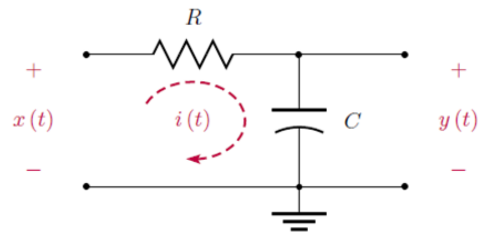
2.4 Constant-Coefficient Ordinary Differential Equations

As an example, the differential equation given by Example 2.4 fits the standard form of Equation with N = 1, M = 0, a1 = 1, a0 = 1/RC, and b0 = 1/RC.

Example 2.4

Differential equation for simple RC circuit

Find a differential equation to describe the input-output relationship for the first-order RC circuit shown.



Solution:

We know that

$$v_R(t) = R\,i(t) \quad \text{and} \quad i(t) = C\,\frac{dy(t)}{dt}$$

Use KVL to obtain

$$RC\,\frac{dy(t)}{dt} + y(t) = x(t) \qquad \Rightarrow \qquad \frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

$$a_N\,\frac{d^N y(t)}{dt^N} + a_{N-1}\,\frac{d^{N-1}y(t)}{dt^{N-1}} + \ldots + a_1\,\frac{dy(t)}{dt} + a_0\,y(t) =$$

$$b_M\,\frac{d^M x(t)}{dt^M} + b_{M-1}\,\frac{d^{M-1}x(t)}{dt^{M-1}} + \ldots + b_1\,\frac{dx(t)}{dt} + b_0\,x(t)$$

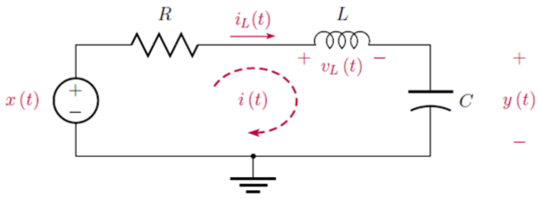
2.4 Constant-Coefficient Ordinary Differential Equations

another example, the differential equation given by Example 2.6 fits the standard form of Equation with N = 2, M = 0, a2 = 1, a1 = R/L, a0 = 1/LC and b0 = 1/LC.

Example 2.6

Differential equation for RLC circuit

Find a differential equation to describe the input-output relationship for the RLC circuit shown.



Solution:

Apply KVL:

$$-x(t) + R\,i(t) + v_L(t) + y(t) = 0$$

$$i(t) = C\,\frac{dy(t)}{dt}\,, \qquad v_L(t) = L\,\frac{di(t)}{dt} = LC\,\frac{d^2y(t)}{dt^2}$$

$$-x(t) + RC\,\frac{dy(t)}{dt} + LC\,\frac{d^2y(t)}{dt^2} + y(t) = 0$$

Rearrange terms:

$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\,\frac{dy(t)}{dt} + \frac{1}{LC}\,y(t) = \frac{1}{LC}\,x(t)$$

$$a_N\,\frac{d^N y(t)}{dt^N} + a_{N-1}\,\frac{d^{N-1}y(t)}{dt^{N-1}} + \ldots + a_1\,\frac{dy(t)}{dt} + a_0\,y(t) =$$

$$b_M\,\frac{d^M x(t)}{dt^M} + b_{M-1}\,\frac{d^{M-1}x(t)}{dt^{M-1}} + \ldots + b_1\,\frac{dx(t)}{dt} + b_0\,x(t)$$